

Integrable Magnetic Model of Two Chains Coupled by Four-Body Interactions

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An exact solution for an XXZ chain with four-body interactions is obtained and its phase diagram is determined. The model can be reduced to two chains coupled by four-body interactions, and it is shown that the ground state of the two-chain model is magnetized in part. Furthermore, a twisted four-body correlation function of the anti-ferromagnetic Heisenberg chain is obtained.

KEYWORDS: quantum inverse scattering method, algebraic Bethe ansatz, thermodynamic Bethe ansatz

§1. Introduction

Quantum spin systems solved exactly so far are essentially one dimensional. It is a challenging problem to find a two, or higher, dimensional exactly solvable quantum spin model. As a first step we study exactly solvable models for two, or more, chains combined each other by some interactions. A well-known example is the Majumdar-Ghosh model,¹⁾ which can be interpreted as a two-leg zigzag ladder;

$$H_{\text{MG}} = \frac{1}{2} \sum_{j=1}^N \mathbf{S}_j \cdot \mathbf{S}_{j+1} + \frac{1}{2} \sum_{j=1}^N \mathbf{T}_j \cdot \mathbf{T}_{j+1} + \sum_{j=1}^N \mathbf{S}_j \cdot \mathbf{T}_j + \sum_{j=1}^N \mathbf{T}_j \cdot \mathbf{S}_{j+1}, \quad (1.1)$$

where \mathbf{S}_j and \mathbf{T}_j are operators of spin- $\frac{1}{2}$ at site j on each chain. The ground states are given by doubly degenerate dimer states with no magnetization.

We introduce here a novel two-chain model, which is exactly solvable by the Bethe ansatz method. The model consists of two Heisenberg chains coupled by four-body interactions. It can be expressed as twisted two chains;

$$H_{2\text{chain}} = \frac{1}{4} \sum_{j=1}^N \mathbf{S}_j \cdot \mathbf{S}_{j+1} + \frac{1}{4} \sum_{j=1}^N \mathbf{T}_j \cdot \mathbf{T}_{j+1} + \sum_{j=1}^N (\mathbf{S}_j \times \mathbf{T}_j) \cdot (\mathbf{S}_{j+1} \times \mathbf{T}_{j+1}) + \sum_{j=1}^N (\mathbf{T}_j \times \mathbf{S}_{j+1}) \cdot (\mathbf{T}_{j+1} \times \mathbf{S}_{j+2}). \quad (1.2)$$

This Hamiltonian is obtained as a special case of the following single-chain model;

$$H_{\Delta, \alpha} = J_1 \sum_{j=1}^N [\mathbf{S}_j \cdot \mathbf{S}_{j+1}]_{\Delta_1} + J_2 \sum_{j=1}^N \mathbf{S}_j \cdot \mathbf{S}_{j+2} + J_3 \sum_{j=1}^N ((\mathbf{S}_j \cdot \mathbf{S}_{j+2})(\mathbf{S}_{j+1} \cdot \mathbf{S}_{j+3}))$$

$$- [\mathbf{S}_j \cdot \mathbf{S}_{j+3}]_{\Delta_3} [\mathbf{S}_{j+1} \cdot \mathbf{S}_{j+2}]_{\Delta'_3}.$$

$$+ \text{const.}, \quad (1.3)$$

where an abbreviation of anisotropic scalar product is used,

$$[\mathbf{a} \cdot \mathbf{b}]_{\Delta} \equiv a^x b^x + a^y b^y + \Delta a^z b^z. \quad (1.4)$$

This model obtains integrability when the anisotropies, Δ_1 , Δ_3 , and Δ'_3 , and the strengths of interactions, J_k , satisfy certain conditions with two parameters, Δ and α .

In the next section the integrability condition of the single-chain model (1.3) is presented. The model is integrable if its Hamiltonian is decomposed to a sum of mutually commuting operators including the XXZ model Hamiltonian. The eigenvalues of the commuting operators are evaluated in §3. The existence of phase transition is discussed in §4, where the phase boundary is determined analytically. The magnetic property of the two-chain model is mentioned briefly in §5. In §6 a correlation function of the nearest four sites of the Heisenberg chain is explicitly given. The last section is devoted to a brief summary.

§2. Integrability

We consider the system of the following Hamiltonian;

$$H_{\Delta, \alpha} = H_{XXZ} + \alpha H_{\text{int}}, \quad (2.1)$$

where the first term on the r. h. s. is the Hamiltonian of the usual anti-ferromagnetic XXZ model,

$$H_{XXZ} = \sum_{j=1}^N [\mathbf{S}_j \cdot \mathbf{S}_{j+1}]_{\Delta} - \frac{\Delta}{4} N, \quad (2.2)$$

and the interaction term, H_{int} , involves up to the third neighbor interactions;

$$H_{\text{int}} = -\frac{1+\Delta^2}{2} \sum_{j=1}^N [\mathbf{S}_j \cdot \mathbf{S}_{j+1}]_{\frac{2\Delta}{1+\Delta^2}} + \frac{\Delta}{2} \sum_{j=1}^N \mathbf{S}_j \cdot \mathbf{S}_{j+2}$$

$$+ 2\Delta \sum_{j=1}^N \left((\mathbf{S}_j \cdot \mathbf{S}_{j+2})(\mathbf{S}_{j+1} \cdot \mathbf{S}_{j+3}) - [\mathbf{S}_j \cdot \mathbf{S}_{j+3}]_{\Delta} [\mathbf{S}_{j+1} \cdot \mathbf{S}_{j+2}]_{\Delta^{-1}} \right) + \frac{\Delta}{8} N. \quad (2.3)$$

The coupling constant, α , takes a real value, the anisotropy parameter is restricted in the range $\Delta > -1$, and the periodic boundary condition, $S_{N+1} = S_1$, is imposed. It is easy to see that the Hamiltonian (2.1) has a similar form to that of the single-chain model (1.3). The former can be considered to be a special case of the latter.

Note that the parameter α does not denote the strength ratio of the second-neighbor, or four-body, interaction to the nearest neighbor one, but expresses only the ratio between H_{int} and H_{XXZ} . The strengths, J_k , and the anisotropies, Δ_k , of the interactions in the Hamiltonian (1.3) should be related with the two parameters, α and Δ , as follow;

$$J_1 = 1 - \frac{1 + \Delta^2}{2} \alpha, \quad J_2 = \frac{\Delta \alpha}{2}, \quad J_3 = 2\Delta \alpha, \quad (2.4)$$

$$\Delta_1 = \frac{2\Delta(1 - \alpha)}{2 - (1 + \Delta^2)\alpha}, \quad \Delta_3 = (\Delta'_3)^{-1} = \Delta. \quad (2.5)$$

The strength ratio of the second-neighbor interaction to the four-body one, J_2/J_3 , is fixed at $1/4$, though J_2/J_1 has freedom.

Conversely, the Hamiltonian (1.3) is integrable under these conditions, (2.4) and (2.5), since it is decomposed to a sum of two commuting Hamiltonians, (2.2) and (2.3). The reason of the commutativity is that the complicated interaction operator H_{int} is introduced by the help of the quantum inverse scattering method^{2,3)} as follows.

The Hamiltonian of the XXZ model is related with the transfer matrix, $\tau(\lambda)$, of the six-vertex model. The transfer matrix is given as the trace of the monodromy matrix;

$$\tau(\lambda) = \text{tr } T(\lambda). \quad (2.6)$$

The monodromy matrix is constructed of L -matrices;

$$T(\lambda) = L_N(\lambda) L_{N-1}(\lambda) \cdots L_1(\lambda). \quad (2.7)$$

The L -matrices are defined as follow;

$$L_j(\lambda) = a(\lambda) \frac{1 + \sigma_{\text{aux}}^z \otimes \sigma_j^z}{2} + d(\lambda) \frac{1 - \sigma_{\text{aux}}^z \otimes \sigma_j^z}{2} + \frac{1}{\sinh \frac{\gamma}{2}(\lambda + i)} (\sigma_{\text{aux}}^+ \otimes \sigma_j^- + \sigma_{\text{aux}}^- \otimes \sigma_j^+), \quad (2.8)$$

with

$$a(\lambda) = \frac{1}{i \sin \gamma}, \quad (2.9)$$

$$d(\lambda) = \frac{1}{i \sin \gamma} \frac{\sinh \frac{\gamma}{2}(\lambda - i)}{\sinh \frac{\gamma}{2}(\lambda + i)}, \quad (2.10)$$

where the parameter γ takes a real or a purely imaginary value, and $\sigma^\pm = \frac{1}{2}(\sigma^x \pm \sigma^y)$. The subscript of the Pauli matrices, $\sigma = (\sigma^x, \sigma^y, \sigma^z)$, denotes the space upon which they act. The trace and the products of L -matrices in the definition of the transfer matrix should be performed

in the auxiliary space of the operator σ_{aux} .

The L -matrices satisfy the Yang-Baxter relation;

$$R(\lambda, \mu) (L(\lambda) \otimes L(\mu)) = (L(\mu) \otimes L(\lambda)) R(\lambda, \mu), \quad (2.11)$$

where the R -matrix is given as

$$R(\lambda, \mu) = f(\lambda, \mu) \frac{1 + \sigma^z \otimes \sigma^z}{2} + g(\lambda, \mu) \frac{1 - \sigma^z \otimes \sigma^z}{2} + \sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+, \quad (2.12)$$

with

$$f(\lambda, \mu) = \frac{\sinh \frac{\gamma}{2}(\lambda - \mu + 2i)}{\sinh \frac{\gamma}{2}(\lambda - \mu)}, \quad (2.13)$$

$$g(\lambda, \mu) = \frac{i \sin \gamma}{\sinh \frac{\gamma}{2}(\lambda - \mu)}. \quad (2.14)$$

Then the transfer matrix forms a commuting family with the spectral parameter λ ;

$$[\tau(\lambda), \tau(\mu)] = 0. \quad (2.15)$$

The k -th order Hamiltonian, $H^{(k)}$, is defined as a logarithmic derivative of the transfer matrix;

$$H^{(k)} = i \left(\frac{\sin \gamma}{\gamma} \frac{\partial}{\partial \lambda} \right)^k \ln \tau(\lambda) \Big|_{\lambda=i}. \quad (2.16)$$

The XXZ model Hamiltonian (2.2) is identical to the Hamiltonian of the first order;

$$H_{XXZ} = H^{(1)}, \quad \Delta = \cos \gamma. \quad (2.17)$$

Because of the commutativity of the transfer matrix, the Hamiltonians of arbitrary order also commute with one another; $[H^{(j)}, H^{(k)}] = 0$. Then they have common eigenstates, which are especially common to those of the XXZ model. A linear combination of the commuting Hamiltonians,

$$H = \sum_{k=1} \alpha_k H^{(k)}, \quad (2.18)$$

is, therefore, integrable by the Bethe ansatz method.

Tsvelik⁴⁾ and Frahm⁵⁾ investigated a model constructed of the first and the second order Hamiltonian;

$$H = H_{XXZ} + \alpha H^{(2)}, \quad (2.19)$$

where the explicit form of the interaction operator is given as

$$H^{(2)} = \Delta \sum_{j=1}^N [\mathbf{S}_j \cdot (\mathbf{S}_{j-1} \times \mathbf{S}_{j+1})]_{\Delta^{-1}}. \quad (2.20)$$

They showed that this model exhibits a phase transition at some finite value of the coupling constant. The critical value, α_{crit} , for arbitrary anisotropy is analytically obtained by Frahm. The ground state at $|\alpha| > \alpha_{\text{crit}}$ obtains an incommensurability and it is also magnetized in the Ising-like region; $\Delta \geq 1$.

The model, however, has the strange interaction (2.20) which breaks the parity invariance. That motivated us to choose the new interaction (2.3), which is nothing but

the third order Hamiltonian;

$$H_{\text{int}} = H^{(3)}. \quad (2.21)$$

This operator conserves the parity. Furthermore, it contains isotropic second-neighbor two-body interactions.

When we take the parameter as $\alpha = 1$ and $\Delta = 1$ in the Hamiltonian (2.1), the nearest neighbor interactions cancel out. It becomes the following form;

$$H_{1,1} = \frac{1}{2} \sum_{j=1}^N \mathbf{S}_j \cdot \mathbf{S}_{j+2} + 2 \sum_{j=1}^N (\mathbf{S}_j \times \mathbf{S}_{j+1}) \cdot (\mathbf{S}_{j+2} \times \mathbf{S}_{j+3}) - \frac{N}{8}. \quad (2.22)$$

This is equivalent with the Hamiltonian of the two-chain model; $H_{1,1} = 2H_{2\text{chains}} - N/8$.

§3. Eigenstates and Eigenvalues

In this section we investigate the system of the Hamiltonian (2.1) by using the algebraic Bethe ansatz method²⁾ and the thermodynamic Bethe ansatz method.^{6, 7, 8, 9, 10)}

We are interested in the ground state of the system. Then we consider the low temperature limit to observe the ground state.

Since the system has the same eigenstates as those of the XXZ model, the states with $S_{\text{total}}^z = (N - 2M)/2$ are determined by the Bethe ansatz equations of the XXZ model. Using the algebraic Bethe ansatz method, we obtain the following equations;

$$\left(\frac{a(\lambda_j)}{d(\lambda_j)} \right)^N = \prod_{l=1, l \neq j}^M \frac{f(\lambda_j, \lambda_l)}{f(\lambda_l, \lambda_j)}, \quad j = 1, 2, \dots, M. \quad (3.1)$$

where the functions $a(\lambda)$, $d(\lambda)$, and $f(\lambda, \mu)$ are given in the eqs. (2.9), (2.10) and (2.13). The functions on the l. h. s., $(a(\lambda))^N$ and $(d(\lambda))^N$, are the eigenvalues of the diagonal elements of the monodromy matrix, $[T(\lambda)]_{1,1}$ and $[T(\lambda)]_{2,2}$, respectively, corresponding to the pseudovacuum state; $|\emptyset\rangle = \prod_{j=1}^N |\uparrow\rangle_j$.

An eigenstate corresponds to a set of the spectral parameters, $\{\lambda_j\}$, which is a solution of the eqs. (3.1). The state vector can be expressed by using the operator of an off-diagonal element of the monodromy matrix;

$$|\{\lambda_j\}\rangle = \prod_{j=1}^M [T(\lambda_j)]_{1,2} |\emptyset\rangle. \quad (3.2)$$

Now we calculate the eigenvalue of the k -th order Hamiltonian (2.16) corresponding to the state (3.2). The eigenvalue of the transfer matrix is calculated as follows;

$$\tau(\mu) |\{\lambda_j\}\rangle = \theta(\mu, \{\lambda_j\}) |\{\lambda_j\}\rangle, \quad (3.3)$$

where

$$\theta(\mu, \{\lambda_j\}) = (a(\mu))^N \prod_{j=1}^M f(\lambda_j, \mu) + (d(\mu))^N \prod_{j=1}^M f(\mu, \lambda_j). \quad (3.4)$$

Taking a logarithmic derivative of this equation, we obtain the eigenvalue of the k -th order Hamiltonian $H^{(k)}$

for $k \geq 1$;

$$E_k(\{\lambda_j\}) = \left(\frac{\sin \gamma}{\gamma} \frac{\partial}{\partial \mu} \right)^{k-1} E(\mu, \{\lambda_j\}) \Big|_{\mu=i}, \quad (3.5)$$

with

$$E(\mu, \{\lambda_j\}) = \frac{i \sin \gamma}{\gamma} \frac{\partial}{\partial \mu} \ln \theta(\mu, \{\lambda_j\}). \quad (3.6)$$

The second term on the r. h. s. of the eq. (3.4) does not contribute to the eigenvalue of the Hamiltonian if $k < N$, since the function $(d(\mu))^N$ and its derivatives vanish at $\mu = i$. We consider the thermodynamic limit, $N \rightarrow \infty$, and the term can be omitted. Then the function (3.6) reduces to

$$E(\mu, \{\lambda_j\}) = \sum_{j=1}^M e(\mu, \lambda_j), \quad (3.7)$$

with

$$e(\mu, \lambda) = \frac{\sin^2 \gamma}{\cos \gamma - \cosh \gamma(\lambda - \mu + i)}. \quad (3.8)$$

The eigenvalue of the k -th order Hamiltonian is given as the derivative of the function (3.7);

$$E_k(\{\lambda_j\}) = \sum_{j=1}^M e_k(\lambda_j), \quad (3.9)$$

with

$$e_k(\lambda) = \left(-\frac{\sin \gamma}{\gamma} \frac{\partial}{\partial \lambda} \right)^{k-1} e(i, \lambda). \quad (3.10)$$

Note that the following identity is used in the derivation of the eq. (3.10);

$$\frac{\partial}{\partial \mu} e(\mu, \lambda_j) = -\frac{\partial}{\partial \lambda_j} e(\mu, \lambda_j). \quad (3.11)$$

The expression (3.9) means that the system is composed of M particles, each of which has rapidity λ , and the energy of the system is given as the sum of one-particle energies (3.10).

§4. Phase Diagram of the Single-Chain Model

We determine the phase diagram of the system (2.1) here in the thermodynamic limit. The energy for the eigenstates (3.2) is given from the results of the previous section;

$$E = \sum_{j=1}^M \left[1 + \alpha \left(\frac{\sinh \gamma}{\gamma} \frac{\partial}{\partial \lambda_j} \right)^2 \right] \frac{\sin^2 \gamma}{\cos \gamma - \cosh \gamma \lambda_j}. \quad (4.1)$$

The parameter γ , which is related with the anisotropy as $\Delta = \cos \gamma$, takes a real value when $|\Delta| \leq 1$ and a purely imaginary value when $\Delta > 1$.

4.1 $|\Delta| < 1$

Here we consider the case that the absolute value of the anisotropy is less than one. The Bethe ansatz equations (3.1) have string solutions^{6, 9)} in the thermodynamic limit $N \rightarrow \infty$;

$$\lambda_j^k = \lambda_j + (n_j - 2k + 1)i + \frac{1 - v_j}{2\gamma} \pi i,$$

$$\text{for } k = 1, \dots, n_j, \quad (4.2)$$

where the length, n_j , of a string is bounded by the number of the particles, $\sum n_j = M$, and the parity, v_j , of a string takes value of ± 1 . The parities originate from the periodicity of the eqs. (3.1) along the imaginary axis. It is thought that the string solutions are sufficient to form thermodynamics.

Following the usual procedure,^{6,9)} we classify the solutions of the Bethe ansatz equations by the lengths and the parities of strings. The density of rapidity and that of holes, $\rho_j(\lambda)$ and $\rho_j^h(\lambda)$, where the subscript denotes a class, is defined. In the thermodynamic Bethe ansatz method, the dressed energy, $\varepsilon_j(\lambda)$, is introduced as

$$\varepsilon_j(\lambda) = T \ln \eta_j(\lambda), \quad \eta_j(\lambda) = \frac{\rho_j^h(\lambda)}{\rho_j(\lambda)}. \quad (4.3)$$

We can deduce a set of functional equations about $\rho_j(\lambda)$, $\rho_j^h(\lambda)$, and $\eta_j(\lambda)$ under the thermodynamic equilibrium condition, but it is lengthy and cumbersome to write down them and they are omitted here. (See refs. 6 and 9 if necessary.)

Though the number of the classes is infinite, in the low temperature limit we only have to consider two of them, both of which belong to the length one, one belongs to the positive parity and the other to the negative, since it is shown by Takahashi and Suzuki⁶⁾ that the energies of the classes belonging to the lengths longer than one are large and that they don't contribute to the ground state. Then we obtain the following functional equations about the densities of rapidity, $\rho_\pm(\lambda)$ and $\rho_\pm^h(\lambda)$, and the dressed energies, $\varepsilon_\pm(\lambda)$, of the two classes;

$$g_\pm(\lambda) = \pm (\rho_\pm(\lambda) + \rho_\pm^h(\lambda)) + K_\pm * \rho_+(\lambda) + K_\mp * \rho_-(\lambda), \quad (4.4)$$

$$\varepsilon_\pm(\lambda) = \varepsilon_\pm^0(\lambda) - K_\pm * \varepsilon_+^{(-)}(\lambda) + K_\mp * \varepsilon_-^{(-)}(\lambda), \quad (4.5)$$

with

$$g_\pm(\lambda) = \pm \frac{1}{2\pi} \frac{\gamma \sin \gamma}{\cosh \gamma \lambda \mp \cos \gamma},$$

$$K_\pm(\lambda) = \pm \frac{1}{2\pi} \frac{\gamma \sin 2\gamma}{\cosh \gamma \lambda \mp \cos 2\gamma},$$

$$\varepsilon_\pm^0(\lambda) = -\frac{2\pi \sin \gamma}{\gamma} \left[1 + \alpha \left(\frac{\sin \gamma}{\gamma} \frac{\partial}{\partial \lambda} \right)^2 \right] g_\pm(\lambda),$$

$$\varepsilon_\pm^{(-)}(\lambda) = \begin{cases} \varepsilon_\pm(\lambda) & \varepsilon_\pm(\lambda) \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the asterisk product $a * b(\lambda)$ denotes a convolution defined as

$$a * b(\lambda) = \int_{-\infty}^{\infty} a(\lambda - \mu) b(\mu) d\mu. \quad (4.6)$$

Note that in the low temperature limit, $T \rightarrow 0$, the following relation holds from the definition (4.3): the rapidity density $\rho(\lambda)$ (the hole density $\rho^h(\lambda)$) is zero when the dressed energy takes a positive (negative) value. The eqs. (4.4) together with this relation become complete to determine the densities of rapidity.

The analytical solutions of the dressed energies are

easily obtained if the coupling constant $|\alpha|$ is sufficiently small. By applying the Fourier transformation to the eqs. (4.5) we obtain

$$\varepsilon_\pm(\lambda) = \mp \frac{\pi \sin \gamma}{2\gamma} \left[1 + \alpha \left(\frac{\sin \gamma}{\gamma} \frac{\partial}{\partial \lambda} \right)^2 \right] f_\pm(\lambda), \quad (4.7)$$

with

$$f_+(\lambda) = \text{sech} \frac{\pi \lambda}{2},$$

$$f_-(\lambda) = \begin{cases} 0 & 0 < \gamma \leq \frac{\pi}{2}, \\ \frac{-\cos(\pi^2/2\gamma) \cosh(\pi \lambda/2)}{\cosh \pi \lambda + \cos(\pi^2/\gamma)} & \frac{\pi}{2} < \gamma < \pi. \end{cases}$$

The solutions (4.7) are valid as long as $\varepsilon_+(\lambda) \leq 0$ and $\varepsilon_-(\lambda) \geq 0$ for arbitrary value of the rapidity λ . (Note that $-\cos(\pi^2/2\gamma)$ is positive when $\pi/2 < \gamma < \pi$.) It is easy to see from the solutions that this condition is satisfied when the coupling constant lies in the finite range, $\alpha_{\text{crit}}^- < \alpha < \alpha_{\text{crit}}^+$, where

$$\alpha_{\text{crit}}^\pm = \pm \left(\frac{2\gamma}{\pi \sin \gamma} \right)^2 h_\gamma^\pm(\pi^2/\gamma), \quad (4.8)$$

with

$$h_\gamma^+(x) = \begin{cases} 1 & 0 < \gamma \leq \frac{\pi}{2}, \\ \frac{1 - \sin^2(x/2)}{1 + \sin^2(x/2)} & \frac{\pi}{2} < \gamma < \pi, \end{cases}$$

$$h_\gamma^-(x) = \begin{cases} 1 & 0 < \gamma \leq \frac{3\pi}{5}, \\ \frac{2 \sin^2 x}{2 \cos^2 x - 4 \cos x + 3} & \frac{3\pi}{5} < \gamma < \pi. \end{cases}$$

The ground state is the same as that of the XXZ model when the coupling constant is in this range, since the Fermi points of the dressed energies are the same. When the coupling constant goes beyond the range, there add new four Fermi points.

The appearance of the new Fermi points is easily observed at the 'free fermion point', $\Delta = 0$, where the XXZ model reduces to a free fermion system. The one-particle dispersion of our model is easily found to be

$$\varepsilon(p) = \left(1 - \frac{\alpha}{2}\right) \cos p - \frac{\alpha}{2} \cos 3p. \quad (4.9)$$

There appear three negative-energy regions with six Fermi points when the coupling constant is beyond the critical value, $|\alpha| > \alpha_{\text{crit}}^\pm = 1$.

Similar situation can be observed on the two dressed energies at another value of the anisotropy parameter. Then we conclude that there occurs a phase transition if the coupling constant goes across the critical value (4.8). The new phase is characterized by added new branches of excitations, which are also gapless.

The ground state in the new phase has no magnetic moment. We can see this fact by taking a Fourier component of one of the eqs. (4.4) at zero frequency. It leads to

$$M/N = S_{\text{total}}^z/N$$

$$\begin{aligned}
&= \frac{1}{2} - \int \rho_+(\lambda) d\lambda - \int \rho_-(\lambda) d\lambda \\
&= \begin{cases} \frac{\pi}{2\gamma} \int \rho_-^h(\lambda) d\lambda & 0 < \gamma \leq \frac{\pi}{2}, \\ \frac{1}{2(1-\gamma/\pi)} \int \rho_+^h(\lambda) d\lambda & \frac{\pi}{2} < \gamma < \pi. \end{cases} \quad (4.10)
\end{aligned}$$

Since it can be proved that $\varepsilon_{-(+)}(\lambda) \leq 0$ everywhere for $0 < \gamma \leq \pi/2$ ($\pi/2 < \gamma < \pi$), there is no hole of rapidity belonging to $-(+)$ -parity. The ground state is, therefore, not magnetized at any strength of the interaction (2.3). The new phase is similar to the corresponding one of the Tsvetlik-Frahm model^{4,5} in this point.

4.2 $\Delta \geq 1$

Here the parameter γ is purely imaginary and we put $\gamma = i\theta$, $\theta \geq 0$. Since in this case there is no periodicity along the imaginary axis in the Bethe ansatz equations (3.1), only string solutions with positive parity, which are centered on the real axis, are allowed. On the other hand, the equations become periodic along the real axis. We restrict, therefore, the rapidities in the following range;

$$\Lambda = \{\lambda \mid -\frac{\pi}{\theta} \leq \lambda \leq \frac{\pi}{\theta}\}. \quad (4.11)$$

(When the parameter γ is zero, an adequate limit, $\theta \rightarrow 0$, should be taken on the equations below in this section and the range of the rapidity should be expanded to infinity; $\Lambda = \{\lambda \mid -\infty < \lambda < \infty\}$.)

We only have to consider string solutions belonging to the length one in the same way as the previous section. The density of rapidity, $\rho(\lambda)$, and the dressed energy, $\varepsilon(\lambda)$, are determined by the following eqs.;

$$g(\lambda) = \rho(\lambda) + \rho^h(\lambda) + K * \rho(\lambda), \quad (4.12)$$

$$\varepsilon(\lambda) = \varepsilon^0(\lambda) - K * \varepsilon^{(-)}(\lambda), \quad (4.13)$$

with

$$g(\lambda) = \frac{1}{2\pi} \frac{\theta \sinh \theta}{\cosh \theta - \cos \theta \lambda},$$

$$K(\lambda) = \frac{1}{2\pi} \frac{\theta \sinh \theta}{\cosh 2\theta - \cos \theta \lambda},$$

$$\varepsilon^0(\lambda) = -\frac{2\pi \sinh \theta}{\theta} \left[1 + \alpha \left(\frac{\sinh \theta}{\theta} \frac{\partial}{\partial \lambda} \right)^2 \right] g(\lambda),$$

$$\varepsilon^{(-)}(\lambda) = \begin{cases} \varepsilon(\lambda) & \varepsilon(\lambda) \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the asterisk product $a * b(\lambda)$ denotes a convolution over the range (4.11) here;

$$a * b(\lambda) = \int_{\Lambda} a(\lambda - \mu) b(\mu) d\mu. \quad (4.14)$$

The eq. (4.13) has an analytical solution for sufficiently small $|\alpha|$. The dressed energy is obtained as

$$\varepsilon(\lambda) = - \left[1 + \alpha \left(\frac{\sinh \theta}{\theta} \frac{\partial}{\partial \lambda} \right)^2 \right] \sum_{n=-\infty}^{+\infty} \frac{1}{2} \text{sech} |n| \theta$$

$$\begin{aligned}
&= -\frac{K(k) \sinh \theta}{\pi} \left[1 + \alpha \left(\frac{\sinh \theta}{\theta} \frac{\partial}{\partial \lambda} \right)^2 \right] \\
&\quad \times \text{dn}(K'(k)\lambda; k), \quad (4.15)
\end{aligned}$$

where the parameters $K(k)$ and $K'(k)$ are the complete elliptic integrals of the first kind with modulus k and $k' = \sqrt{1-k^2}$, respectively, and the function $\text{dn}(x; k)$ is one of Jacobi's elliptic function with modulus k . The modulus is related with the anisotropy as

$$\Delta = \cosh \theta, \quad \theta = \pi \frac{K'(k)}{K(k)}. \quad (4.16)$$

The solution (4.15) is valid as long as $\varepsilon(\lambda) \leq 0$ everywhere. This condition is identical with the following one;

$$|\alpha| < \alpha_{\text{crit}} = \left(\frac{\theta}{k K'(k) \sinh \theta} \right)^2. \quad (4.17)$$

The critical value is continuous at $\gamma = 0$ with that of the previous section taking value of $\pm(2/\pi)^2$.

The ground state coincides with that of the XXZ model when the condition (4.17) is satisfied. It is not magnetized. If the anisotropy is in the 'Ising-like region', $\Delta > 1$, the ground state also shows the Néel order and has excitations with gap.

When the coupling constant is beyond the critical value, the dressed energy gains a positive region where $\varepsilon(\lambda) > 0$. We name the region D^+ , and define its complementary set D^- ;

$$D^+ = \{\lambda \in \Lambda \mid \varepsilon(\lambda) > 0\}, \quad D^- = \Lambda - D^+. \quad (4.18)$$

The existence of not-empty region D^+ means that there appear new Fermi points and the system has gapless excitations, since the dressed energy function, $\varepsilon(\lambda)$, is continuous and takes a definite value at the end points of domain Λ . Thus the system shows a phase transition at the critical value of the coupling constant. When the anisotropy parameter lies in the 'Ising-like region', the system undergoes a transition from the gapful phase to the gapless one when the coupling constant exceeds the critical value.

Another typical character of the new phase is that the ground state is magnetized. We can calculate the magnetic moment as

$$\begin{aligned}
M/N &= S_{\text{total}}^z / N \\
&= \frac{1}{2} - \int_{\Lambda} \rho(\lambda) d\lambda \\
&= \frac{1}{2} \int_{\Lambda} \rho^h(\lambda) d\lambda, \quad (4.19)
\end{aligned}$$

by taking the zero frequency component of the Fourier expansion of the eq. (4.12). As we noted in the previous section, the hole density, $\rho^h(\lambda)$, can have non zero value only in the region D^+ . When the coupling constant is large enough for the region D^+ not to be empty, it is easy to make sure from the eq. (4.12) that there exist holes. Thus the last term of the eq. (4.19) gives a positive value and the system obtains a magnetic moment.

Note that the system is not completely but partly magnetized even for quite large coupling constant, since the

region D^- never becomes empty. In fact the magnetic moment gets saturated at $S_{\text{total}}^z/N = 0.3217$ (0.1479) when $\alpha \rightarrow +\infty$ ($-\infty$) and $\Delta = 1$.

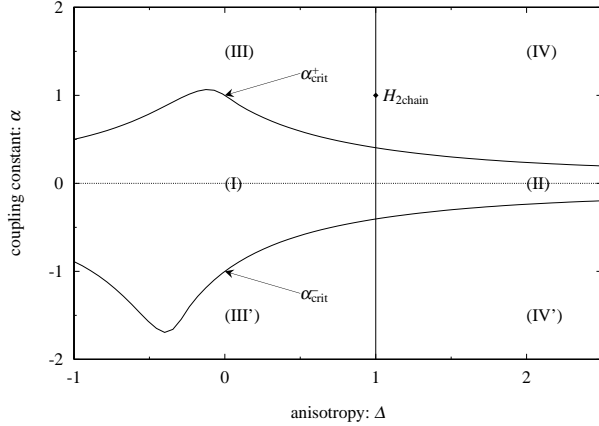


Fig. 1. Phase diagram of the single-chain model (2.1) is given. Its ground state has the same property as that of the XXZ model for sufficiently small $|\alpha|$ between α_{crit}^+ and α_{crit}^- ; the phase (I), $\Delta \leq 1$, has gapless excitations, the phase (II), $\Delta > 1$, has excitations with gap and the ground states in both phases are not magnetized. The systems in the region (III) and (IV) obtain additional branches of excitation to those in phase (I) and (II), respectively. The extra excitations in both phases are gapless ones. The ground state in the phase (III) is also not magnetized, though in phase (IV), $\Delta \geq 1$, it obtains a finite, but not-full, magnetization. When the parameters are chosen as $\Delta = \alpha = 1$, the model reduces to the two-chain model, which lies in the partially magnetized phase (IV').

§5. Two-Chain Model

Now we can get information of the two-chain model. We only have to watch the single-chain model (2.1) at a special value of the parameter, $\alpha = 1$ and $\Delta = 1$. From the view point of the single-chain model, the coupling constant is beyond the critical value, $\alpha_{\text{crit}} = (2/\pi)^2 < 1$. Then the system of the two-chain model stays in the partly magnetized phase.

The magnetic moment is calculated numerically;

$$S_{\text{total}}^z/N = 0.1049, \quad (5.1)$$

which shows that the system is about 20% magnetized in the ground state. As noted in the previous section, the system has gapless excitations with extra branches.

§6. Twisted Four-Body Correlation Function

The expectation value of the k -th order Hamiltonian operator, $H^{(k)}$, for the ground state of the isotropic and anti-ferromagnetic Heisenberg model (XXX model) is presented in this section. We evaluate it by using the eq. (3.9) and the rapidity distribution function, $\rho_0(\lambda)$, of the ground state of the XXX model;

$$\begin{aligned} \rho_0(\lambda) &= \frac{1}{4} \text{sech} \frac{\pi \lambda}{2} \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \text{sech} \omega \exp(-i\lambda\omega) d\omega. \end{aligned} \quad (6.1)$$

Note that in this section we analyze only the isotropic case, $\Delta = 1$.

The expectation value, $\langle H^{(k)} \rangle_0$, is calculated from the eq. (3.9) in an analytical form;

$$E_k = \frac{\langle H^{(k)} \rangle_0}{N}. \quad (6.2)$$

This vanishes if the integer k is even, and it gives the following value for odd k ;

$$\begin{aligned} E_k &= \int_{-\infty}^{\infty} e_k(\lambda) \rho_0(\lambda) d\lambda \\ &= \begin{cases} (-1)^{(k+1)/2} 2^{-k+1} (1 - 2^{-k+1}) \\ \quad \times (k-1)! \zeta(k) & \text{for } k \geq 3, \\ -\ln 2 & \text{for } k = 1, \end{cases} \end{aligned} \quad (6.3)$$

where the function $\zeta(s)$ is Riemann's zeta, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

The expectation value of the first order Hamiltonian leads the nearest neighbor correlation function of the XXX model;

$$\begin{aligned} \langle \mathbf{S}_j \cdot \mathbf{S}_{j+1} \rangle_0 &= \frac{\langle H^{(1)} \rangle_0}{N} + \frac{1}{4} \\ &= \frac{1}{4} - \ln 2. \end{aligned} \quad (6.4)$$

Similarly the expectation value of the two-chain model Hamiltonian (2.22) gives the following relation;

$$\begin{aligned} \frac{1}{2} \langle \mathbf{S}_j \cdot \mathbf{S}_{j+2} \rangle_0 + 2 \langle (\mathbf{S}_j \times \mathbf{S}_{j+1}) \cdot (\mathbf{S}_{j+2} \times \mathbf{S}_{j+3}) \rangle_0 \\ = \frac{1}{8} - \ln 2 + \frac{3}{8} \zeta(3). \end{aligned} \quad (6.5)$$

The correlation function between the second-neighbor sites is obtained by one of the authors;¹¹⁾

$$\langle \mathbf{S}_j \cdot \mathbf{S}_{j+2} \rangle_0 = \frac{1}{4} - 4 \ln 2 + \frac{9}{4} \zeta(3). \quad (6.6)$$

Then the twisted four-body correlation function is calculated as

$$\begin{aligned} \langle (\mathbf{S}_j \times \mathbf{S}_{j+1}) \cdot (\mathbf{S}_{j+2} \times \mathbf{S}_{j+3}) \rangle_0 &= \frac{1}{2} \ln 2 - \frac{3}{8} \zeta(3) \\ &= -0.104198, \end{aligned} \quad (6.7)$$

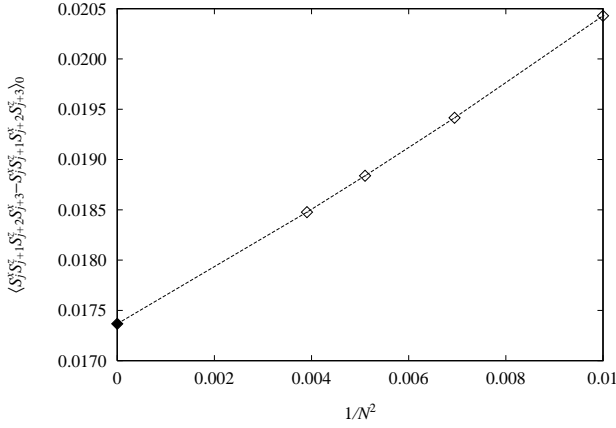
or expressed by its component as

$$\begin{aligned} \langle S_j^\alpha S_{j+1}^\beta S_{j+2}^\alpha S_{j+3}^\beta - S_j^\alpha S_{j+1}^\beta S_{j+2}^\beta S_{j+3}^\alpha \rangle_0 \\ = \left(\frac{1}{16} \zeta(3) - \frac{1}{12} \ln 2 \right) (1 - \delta_{\alpha\beta}) \\ = 0.0173663 (1 - \delta_{\alpha\beta}), \quad \alpha, \beta = x, y, z. \end{aligned} \quad (6.8)$$

To check the validity of this result, we calculate the four-body correlation function for finite systems using the exact diagonalization method and plot them. Figure. 2 supports that the eq. (6.8) is correct.

§7. Summary

The XXZ model with competing four-body interaction has been studied. When the four-body interaction is weaker than the critical value, α_{crit} , the ground state



[11] M. Takahashi: J. Phys. **C10** (1977) 1289.

Fig. 2. The twisted four-body correlation functions of the anti-ferromagnetic Heisenberg chain with N sites, $N = 10, 12, 14$, and 16 , are plotted as a function of $1/N^2$. In the thermodynamic limit, $N \rightarrow \infty$, these values approach the theoretical value (6.8).

coincides with that of the XXZ model and the spectrum of low-lying excitation is also the same. If it gets stronger than α_{crit} , the new phases arise. The critical value of the coupling constant has been obtained analytically. The new phases (III) and (IV), see Fig. 1, have gapless excitations. The phase (III), $|\Delta| < 1$, the ground state is not magnetized and the phase (IV), $\Delta \geq 1$, it is magnetized in part.

The two-chain model (1.2) coupled by four-body interactions can be deduced from the above single-chain model. Since the former model corresponds to the latter in the phase (IV), the ground state of the two-chain model shows the partly magnetized property and has excitations without gap.

From the expectation value of the four-body interaction operator, the twisted four-body correlation of the anti-ferromagnetic Heisenberg model has also been obtained analytically.

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